CONVECTIVE DIFFUSION IN FISSURED-POROUS MEDIA

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The mixing of a dynamically neutral admixture added to a stream flowing through a homogeneous porous medium is described by an equation of the diffusion type with some effective diffusion coefficient which varies linearly with the filter velocity in the flow region in which Darcy's law is obeyed [1]. According to the ideas developed in a whole series of papers [2-4] this process, also called convective diffusion, is due to the irregular nature of the porous canals through which the liquid moves. Molecular effects also play a definite role in the mechanism of mixing, and their relative contribution is greater, the lower the filter velocity.

This paper proposes equations for convective diffusion in fissured-porous media with due regard to the specific nature of mixing in these media. The solutions of some problems are given.

1. Equations of convective diffusion in fissured-porous media. According to [5], fissured-porous rock is a continuous medium consisting of two systems—a system of fissures and a system of blocks enclosing one another. An exchange of liquid takes place between these systems.

Factual data relating to fissured strata indicate that the permeability k_1 of the fissures is several orders greater than the permeability k_2 of the blocks, but the porosity m_1 of the fissure system is much less than the porosity m_2 of the blocks. It is characteristic of fissured rocks that the liquid flows mainly through the fissures, since the filter velocity through the blocks is negligibly small in comparison with the filter velocity through the fissures.

In [3] the effective coefficient of convective diffusion in an ordinary porous medium was represented in the form $D = \lambda u + D_0$ (for the plane unidimensional case) where D_0 is the coefficient of molecular diffusion ($D_0 \sim 10^{-5}$ cm^2/sec), u is the mean flow velocity, and λ is the coefficient of longitudinal dispersion ($\lambda \sim 0.1$ cm).

As estimates show, when the filtration parameters of the blocks have average values ($k_2 \sim 1$–10 μD, $m_2 \sim 0.1$, viscosity of liquid $\mu \sim 1$–10 cp, pressure drop $\Delta p/\Delta x \sim 0.1$–$1$ atm/m) the value of D in the blocks has the order of the coefficient of molecular diffusion D_0. At the same time, in the system of fissures, regarded as a separate porous medium, $\lambda u \gg D_0$, i.e., the effect of molecular transport through the fissures can be neglected.

Thus, the main feature of the mixing of a dynamically neutral admixture in fissured rocks is that convective mixing, which is due to the disordered nature of the fissure system and porous channels of the blocks and depends on the mean flow velocity, plays a significant role only in the fissure system. In weakly permeable blocks diffusion will be due to a molecular type of mechanism.

In view of the essentially different conditions of mixing in the fissures and blocks it is logical to introduce at each point in space two concentrations of diffusing substance: C_1 and C_2. Concentration C_1 and C_2 are the mean concentrations of the admixture in the fissures and pores of the blocks, respectively, in the vicinity of the particular point.

A characteristic feature of mixing in the considered medium is the presence of a flow of the diffusing substance between the fissures and blocks due to the difference in concentrations in the fissure and block systems. Denoting by q the amount of diffusing substance passing from the blocks into the fissures in unit time per unit volume of rock we write the equation of material balance in the fissure system as

$$m_1 \frac{\partial C_1}{\partial t} - \operatorname{div}(D_{ij} \operatorname{grad} C_1 - \mathbf{V} C_1) - q = 0. \tag{1.1}$$

Here **V** and D_{ij} are the filter velocity and the coefficient of convective diffusion, respectively, in the fissure system. Neglecting molecular diffusion, we follow [3] and put D_{ij} in the form

$$D_{ij} = (\lambda_1 - \lambda_2) \mid \mathbf{V} \mid \delta_{ij} + \lambda_2 v_i v_j / \mid \mathbf{V} \mid .$$

Here λ_1 and λ_2 are the coefficients of longitudinal and transverse dispersion in the fissure system; v_i and v_j are the components of the filter velocity of the flow; δ_{ij} is the Kronecker delta.

Neglecting the transfer of admixture through the blocks due to diffusion and convection, we obtain the equation of material balance in the block system:

$$m_2 \partial C_2 / \partial t + q = 0. \tag{1.2}$$

We will confine ourselves henceforth to the case of a steady filtration flow and, in accordance with the above estimates, we will assume that the exchange between the fissure and block systems is effected by the mechanism of molecular diffusion.

The expression for the specific counterflow q depends significantly on the ratio of the characteristic time T of the process and the characteristic time for establishment of a quasistationary distribution of concentration in a single block $\tau_0 \sim L^2/D_0$, where L is the mean dimension of a block.

We consider the two limiting cases in which $\tau_0 \ll T$ and $\tau_0 \gg T$. If $\tau_0 \ll T$, the distribution of concentration within the blocks at any instant is close to the equilibrium level, and for the counterflow rate q we can use the expression

$$q = \alpha (C_2 - C_1) \tag{1.3}$$

which is an exact analogy of the expressions used in [5, 6] for the heat and mass flows in the description of heat-transfer processes in heterogeneous media and filtration in fissured rocks.

The coefficient α has the dimension of inverse time and depends on: 1) the coefficient of molecular diffusion D_0; 2) the geometric parameters of the medium, which determine the area of contact of the liquid particles present in the blocks and fissures (in unit volume of rock). As such parameters we can take the voidage (porosity) m_2 of the blocks and the specific surface σ of the fissures, i.e., the friction surface per unit volume of rock. The value of D_0 is proportional to the coefficient of "free" molecular diffusion D^0 and depends, generally speaking, on the microstructure of the rock. Assuming that the values of D_0 and D^0 are of the same order and using dimensional analysis, we obtain the estimate

$$\alpha \sim m_2 \sigma^2 D^0.$$

We note that in this case, where the contribution of the convective mechanism is commensurable with the diffusion mechanism, the coefficient α can be put in the form

$$\alpha \sim m_2 \sigma^2 D^0 + \frac{k_2 \sigma}{\mu} \mid \operatorname{grad} p \mid . \tag{1.4}$$

The second term in (1.4) takes into account the convective component of the counterflow when the pressure distribution is steady.

As estimates show, for the usual values of the diffusion coefficient in liquids at moderate temperatures ($D_0 \sim 10^{-5}$–$4 \cdot 10^{-5}$ cm^2/sec) the considered case of diffusion with a quasistationary form of counterflow (1.3) can occur only when the blocks are sufficiently small ($L \sim 10$ cm), but when D_0 has larger values, which can be encountered at high temperatures or in the case of gas diffusion, the range of applicability of relationship (1.3) is wider, of course.

We consider now the second case ($\tau_0 \gg T$), which occurs when the blocks have $L \gg 50$–100 cm and is more interesting from the practical viewpoint. The transfer process in this case is essentially unsteady and the expression for q can be obtained, of course, from a consideration of the problem of diffusion in an individual block, in much the same way as was done in the description of capillary impregnation of blocks in a fissured-porous medium [1, 7].

We will confine ourselves initially to the times when the effect of the finite dimensions of the block can be neglected and will consider as a model the molecular diffusion in a linear element of a block on the surface of which the concentration of admixture is equal to the concentration C_1 in the fissure system. We note that this assumption, which greatly simplifies the expression for q, is not too restrictive for the case of sufficiently large blocks, since the effect of the boundaries of the block becomes appreciable at times comparable with the characteristic time of the process.

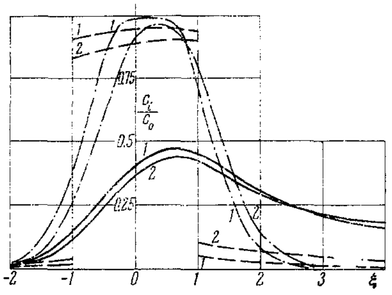

Fig. 1

Proceeding from the known solution of the one-dimensional heat-conduction equation with a condition of the first kind at the boundary of a semi-infinite rod [8], and calculating the flow at this boundary, we obtain an expression for q in the form

$$q = -a \frac{\partial}{\partial t} \int_0^t \frac{(C_1 - C_0)\, d\tau}{\sqrt{\pi (t-\tau)}}, \tag{1.5}$$

where C_0 is the initial distribution of concentration, and for coefficient a, using dimensional analysis, we obtain the estimate

$$a \sim m_2 \sigma \sqrt{D^0}.$$

We note that a similar expression for the rate of counterflow of a mass of liquid was used in [9], where the initial moment of transient processes of pressure redistribution in a homogeneous liquid in a fissured-porous medium was considered.

For an approximate consideration of the effect of the boundaries of the block we will represent the block by a rod of length $2l$ or a sphere of radius l.

In these cases we obtain an expression for q in the form

$$q = -a \frac{\partial}{\partial t} \int_0^t \frac{(C_1 - C_0)\, \varphi(t-\tau)}{\sqrt{\pi (t-\tau)}}\, d\tau, \tag{1.6}$$

where for the case of the rod

$$\varphi(t) = \theta_4(0, \varkappa), \tag{1.7}$$

and for the case of the sphere

$$\varphi(t) = \theta_3(0, \varkappa) - \varkappa^{-1/2}, \tag{1.8}$$

where $\varkappa = l^2/\pi D_0 t$, θ_3 and θ_4 are theta functions. It is easy to show that when $\varkappa \gg 1$, which corresponds to the initial stage of the process, we obtain from (1.7) and (1.8), respectively, $\varphi(t) \sim 1 - 2\exp(-\pi\varkappa) \sim 1$ and $\varphi(t) \sim 1 - \varkappa^{-1/2} \sim 1$, i.e., formula (1.6) agrees with (1.5), and when $\varkappa \sim 0$ for both cases $\varphi(t) \sim 2\varkappa^{-1/2} \exp(-\pi/4x) \sim 0$.

Proceeding from relationships (1.5)–(1.8) we can infer that for real media the expression for the rate of diffusion counterflow between the blocks and fissures can be represented in the form (1.6) (proposed as a result of joint discussion with B. V. Shalimov) and the dimensionless function $\varphi(t)$, which is independent of the law of variation of C_1 and decreases monotonically from unity to zero, will have to be determined from experiments on "diffusion impregnation" of blocks by a method similar to that used by the authors of [10] to investigate capillary impregnation in a fissured-porous medium.

We will dwell briefly on the special features of the formulation and solution of problems in the considered cases. The system of equations (1.1), (1.2), and (1.3) is very similar to the equations of the elastic filtration regime in a fissured-porous medium [5].

It is convenient to solve the problems for this system by eliminating one of the unknown functions and formulating initial and boundary conditions in terms of the required quantity. In particular, if we adopt the method used in [5] it is easy to show that in this case the discontinuities of the concentration C_1 and its normal derivatives $\partial C_1/\partial n$ disappear instantaneously, and for the discontinuities of C_2 and $\partial C_2/\partial n$ we have the relationships

$$[C_2] = [C_2]_{t=0} \exp\left(-\frac{\alpha}{m_2} t\right),$$

$$\left[\frac{\partial C_2}{\partial n}\right] = \left[\frac{\partial C_2}{\partial n}\right]_{t=0} \exp\left(-\frac{\alpha}{m_2} t\right), \tag{1.9}$$

where n is the normal to the fracture surface, and the sign [ ] denotes the discontinuity of the quantity.

When the system of equations (1.1), (1.2), (1.5)–(1.8) is used, it is natural to determine C_1 first and then to find C_2 from (1.2) by quadrature. The distribution of C_1 is continuous and the law of diminution of the discontinuity of concentration in the blocks follows from the explicit form of the solution for C_2.

2. Some problems of convective diffusion in a fissured-porous medium. 1. We consider the solution of the unidimensional problem of mixing of an interlayer of colored liquid with other liquid moving through a fissured-porous medium. We will proceed from the system of equations (1.1)–(1.4) and assume for simplicity that $m_1 = 0$.

Let the colored liquid with concentration C_0 at the initial instant occupy the region of a rectilinear fissured-porous stratum $x_1 \le x \le x_2$. Outside this interlayer the concentration of admixture at t = 0 is zero.

The problem reduces to solution of the equation

$$b \frac{\partial^3 C_2}{\partial \xi^2 \partial \tau} + \frac{\partial^2 C_2}{\partial \xi^2} - 2b \frac{\partial^2 C_2}{\partial \xi\, \partial \tau} - 2\frac{\partial C_2}{\partial \xi} - \frac{\partial C_2}{\partial \tau} = 0$$

$$\left(\xi = \frac{vx}{2D},\ \tau = \frac{v^2 t}{4 m_2 D},\ b = \frac{v^2}{4\alpha D},\ D = \lambda v\right) \tag{2.1}$$

with the initial discontinuity condition

$$C_2(\xi, 0) = C_0 f(\xi), \qquad f(\xi) = \eta(\xi - \xi_1) - \eta(\xi - \xi_2). \tag{2.2}$$

Here v is the filter velocity and $\eta(\xi)$ is a Heaviside function.

As was shown above, the initial discontinuities of the function $C_2(\xi, \tau)$ do not disappear instantaneously, but diminish according to law (1.9). Thus, the required solution, understood as generalized in S. L. Sobolev's sense, is a piecewise-continuous function with discontinuities of the first kind. It is convenient to seek the solution of problem (2.1) and (2.2) in the form

$$C_2(\xi, \tau) / C_0 = u(\xi, \tau) + \exp(-\tau / b) f(\xi),$$

where $u(\xi, \tau)$ is a sufficiently smooth function, which, as can easily be shown, satisfies the equation

$$b \frac{\partial^3 u}{\partial \xi^2 \partial \tau} + \frac{\partial^2 u}{\partial \xi^2} - 2b \frac{\partial^2 u}{\partial \xi\, \partial \tau} - 2\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \tau} + \frac{1}{b}\exp(-\tau/b) = 0 \tag{2.3}$$

and zero initial condition $[u(\xi, 0) \equiv 0]$. Applying the Fourier transformation in variable ξ to (2.3) we obtain for the transformant $U(\nu, \tau)$ the equation

$$(b\nu^2 + 2b\nu i + 1)\, dU / d\tau + \nu(\nu + 2i)\, U = F(\nu, \tau),$$

$$U(\nu, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(\xi, \tau)\, e^{-i\nu\xi}\, d\xi,$$

$$F(\nu, \tau) = \frac{i \exp(-\tau / b)}{\sqrt{2\pi}\, b\nu} \left(e^{-i\nu\xi_2} - e^{-i\nu\xi_1}\right) \tag{2.4}$$

with initial condition $U(\nu, 0) = 0$. Determining $U(\nu, \tau)$ and using the Fourier transformation formula, we obtain the solution in the form

$$C_2(\xi,\tau) = [\eta(\xi-\xi_1) - \eta(\xi-\xi_2)]e^{-\tau/b} +$$
$$+\frac{2}{\pi}\int_0^\infty \Big\{ e^{-\tau/b} \cos\frac{2\xi-\xi_1-\xi_2}{2}\nu -$$
$$-\exp\left[-\frac{b\nu^4+(1+4b)\nu^2}{(1+b\nu^2)^2+4b^2\nu^2}\tau\right]\cos\left[\frac{2\xi-\xi_1-\xi_2}{2}\nu -\right.$$
$$\left.-\frac{2\tau\nu}{(1+b\nu^2)^2+4b^2\nu^2}\right]\Big\} \times \sin\frac{\nu(\xi_1-\xi_2)}{2}\,\frac{d\nu}{\nu}. \qquad (2.5)$$

The distribution for the concentration in the fissures will be a continuous function and is found from Eq. (1.2).

Expanding the exponential functions contained in the formulas for C_1 and C_2 in a series and retaining terms of the order τ, we obtain a representation of the concentration in the fissures and blocks for small τ. Without writing out the obtained expressions, we give the results of the calculations.

Figure 1 shows graphs of the functions $C_1(\xi,\tau)$ (solid lines), $C_2(\xi,\tau)$ (dashed) for the case $b = 1$, $\xi_1 = -1$, $\xi_2 = +1$, and $\tau = 0.1$ (curves 1) and $\tau = 0.2$ (curves 2). The same figure shows the curves (dot-dash lines) calculated from the known solution of the analogous problem in an ordinary porous medium.

1. We consider now the problem of propagation of an admixture in a plane-radial steady flow, where in a well of radius r_0, tapping an initially "pure" infinite layer, a constant concentration C° of admixture is maintained.

Using the system of equations (1.1), (1.2), and (1.5), and putting $m_1 = 0$, we obtain the boundary-value problem for the determination of C_1:

$$\lambda\frac{\partial^2 C_1}{\partial r^2} - \frac{\partial C_1}{\partial r} - \beta r\frac{\partial}{\partial t}\int_0^t \frac{C_1\,d\tau}{\sqrt{\pi(t-\tau)}} = 0 \qquad \left(\beta = \frac{2\pi h a}{Q}\right)$$
$$C_1(r_0, t) = C^\circ, \qquad C_1(t,\infty) = 0. \qquad (2.6)$$

Here Q is the output of the well and h is the thickness of the stratum. Applying the Laplace transformation to (2.6) we obtain an equation for the image of U (y, s),

$$\frac{d^2U}{dy^2} - \frac{dU}{dy} - \gamma y U = 0, \qquad U(y,s) = \int_0^\infty C_1(y,t)\,e^{-st}\,dt$$
$$U(y_0, s) = C^\circ/s, \quad U(\infty, s) = 0,$$
$$y = r/\lambda, \quad y_0 = r_0/\lambda, \quad \gamma = \beta\lambda^2\sqrt{s}. \qquad (2.7)$$

Putting $U = \exp(0.5\,y)\,V$, $z = 0.25\,(1 + 4\gamma y)$ we bring (2.7) to the form

$$\frac{d^2V}{dz^2} - \frac{z}{\gamma^2}V = 0. \qquad (2.8)$$

The general solution of (2.8) has the form

$$V(z,s) = \sqrt{z}\left[AJ_{1/3}\left(\frac{2i}{3\gamma}z^{3/2}\right) + BY_{1/3}\left(\frac{2i}{3\gamma}z^{3/2}\right)\right].$$

Here $J_{1/3}$ and $Y_{1/3}$ are standard symbols of Bessel functions; A and B are arbitrary constants. Reverting to the initial variables and using the boundary conditions, we obtain the solution of problem (2.7), (2.8) after some transformations in the form

$$U(y,s) = \frac{C^\circ}{s}\exp\left(\frac{y-y_0}{2}\right)\left(\frac{1+4\gamma y}{1+4\gamma y_0}\right)^{1/2}\frac{K_{1/3}(\theta(y))}{K_{1/3}(\theta(y_0))}$$
$$\left(\theta(y) = \frac{(1+4\gamma y)^{3/2}}{12\gamma}\right). \qquad (2.9)$$

Here $K_{1/3}$ is the symbol of a Macdonald function.

In the general case transformation (2.9) leads to a fairly difficult expression. Estimates show, however, that in the range of real values of the parameters ($Q \geqslant 50$ m^3/day, $m_2 \sim 0.2$, $h \sim 10$m, $L \sim 100$ cm), $r \leqslant 500$ m, and for not very small values of time the argument of functions $K_{1/3}$ is large, and in this case, putting $y_0 = 0$ for simplicity, we obtain an expression for U in the form

$$U(y,s) = C^\circ s^{-1}(1-\gamma y)\exp(-0.5\,\gamma y^2). \qquad (2.10)$$

Transforming (2.10), we find the solution for C_1 in the form

$$\frac{C_1(\zeta,\eta)}{C^\circ} = \operatorname{erfc}\eta - \frac{2}{\sqrt{\pi}}\zeta\sqrt{\eta}\exp(-\eta^2)$$
$$\left(\zeta = \frac{\lambda\sqrt{\beta}}{\sqrt[4]{t}}, \quad \eta = \frac{\beta r^2}{4\sqrt{t}}\right). \qquad (2.11)$$

Using (1.2), we obtain

$$\frac{C_2(\zeta,\eta)}{C^\circ} = \frac{\psi}{\zeta^2}[\exp(-\eta^2) - \sqrt{\pi\eta}(\zeta+\sqrt{\eta})\operatorname{erfc}\eta]$$
$$\left(\psi = \frac{2a\beta\lambda^2}{\sqrt{\pi}m_2}\right). \qquad (2.12)$$

Figure 2 gives the results of calculations from formulas (2.11) and (2.12) for the case $\psi = 10^{-7}$ and $\zeta = 10^{-3}$ (curves 1), $\zeta = 5.10^{-4}$ (curves 2).

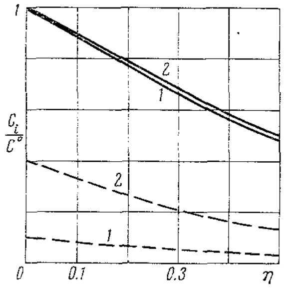

Fig. 2

As the considered problems show, the characteristic feature of the mechanism of convective diffusion in a fissured-porous medium is the relatively rapid propagation of admixture through the fissures and the very appreciable retardation of this process in weakly permeable blocks.

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